

Answers to Exercise 2

1. Find the first, second and cross partial derivatives of the following functions.

$$i) \quad f(x, z) = u = 4x^2 - 6xz + 4z^2 - 3x - 3z$$

$$\frac{\partial u}{\partial x} = 8x - 6z - 3, \quad \frac{\partial u}{\partial z} = -6x + 8z - 3$$

$$\frac{\partial^2 u}{\partial x \partial x} = 8, \quad \frac{\partial^2 u}{\partial z \partial z} = 8$$

$$\frac{\partial^2 u}{\partial x \partial z} = -6, \quad \frac{\partial^2 u}{\partial z \partial x} = -6$$

$$ii) \quad f(x, z) = u = 2x^3 - 6x^2z^2 + 4z^3 - 3xz - z - x$$

$$\frac{\partial u}{\partial x} = 6x^2 - 12xz^2 - 3z - 1, \quad \frac{\partial u}{\partial z} = -12x^2z + 12z^2 - 3x - 1$$

$$\frac{\partial^2 u}{\partial x \partial x} = 12x - 12z^2, \quad \frac{\partial^2 u}{\partial z \partial z} = -12x^2 + 24z$$

$$\frac{\partial^2 u}{\partial x \partial z} = -24xz - 3, \quad \frac{\partial^2 u}{\partial z \partial x} = -24xz - 3$$

$$iii) \quad f(x, y, z) = u = x^2 - y^2 - 4z^2 - 3xyz - 3zy - 2xz - xy$$

$$\frac{\partial u}{\partial x} = 2x - 3zy - 2z - y, \quad \frac{\partial u}{\partial z} = -8z - 3xy - 3y - 2x, \quad \frac{\partial u}{\partial y} = -2y - 3xz - 3z - x$$

$$\frac{\partial^2 u}{\partial x \partial x} = 2, \quad \frac{\partial^2 u}{\partial z \partial z} = -8, \quad \frac{\partial^2 u}{\partial y \partial y} = -2$$

$$\frac{\partial^2 u}{\partial x \partial z} = \frac{\partial^2 u}{\partial z \partial x} = -3y - 2, \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = -3z - 1, \quad \frac{\partial^2 u}{\partial z \partial y} = \frac{\partial^2 u}{\partial y \partial z} = -3x - 3$$

2. Find the turning point(s) of the first function in question one, and determine whether they are minimum or maximum.

$$i) \quad f(x, z) = u = 4x^2 - 6xz + 4z^2 - 3x - 3z$$

$$\frac{\partial u}{\partial x} = 8x - 6z - 3 = 0$$

$$\frac{\partial u}{\partial z} = -6x + 8z - 3 = 0 \Rightarrow x = 1.5 \quad z = 1.5$$

$$\frac{\partial^2 u}{\partial x \partial x} = 8 > 0, \quad \frac{\partial^2 u}{\partial z \partial z} = 8 > 0, \quad \frac{\partial^2 u}{\partial x \partial z} = -6, \quad \frac{\partial^2 u}{\partial z \partial x} = -6$$

$$\frac{\partial^2 u}{\partial x \partial z} * \frac{\partial^2 u}{\partial z \partial x} > \left(\frac{\partial^2 u}{\partial x \partial z} \right)^2, \quad 64 > 36 \therefore \text{we have a minimum}$$

3. If the profit function of a company producing two types of goods is,

$$P = 25x - x^2 - xz - 2z^2 + 30z - 28$$

find, i) the combination of x and z that maximises the profit.
ii) the maximum level of profit at this output combination.

i) $f(x, z) = P = 25x - x^2 - xz - 2z^2 + 30z - 28$

$$\frac{\partial u}{\partial x} = -2x - z + 25 = 0$$

$$\frac{\partial u}{\partial z} = -4z - x + 30 = 0 \Rightarrow x = 10 \quad z = 5$$

$$\frac{\partial^2 u}{\partial x \partial x} = -2 < 0, \quad \frac{\partial^2 u}{\partial z \partial z} = -4 < 0, \quad \frac{\partial^2 u}{\partial x \partial z} = -1, \quad \frac{\partial^2 u}{\partial z \partial x} = -1$$

$$\frac{\partial^2 u}{\partial x \partial x} * \frac{\partial^2 u}{\partial z \partial z} > \left(\frac{\partial^2 u}{\partial x \partial z} \right)^2, \quad 16 > 1 \therefore \text{we have a maximum}$$

Maximum profit at this productiuon level is = 172

4. If the demand function for a commodity is $Q=700-2P+0.02I$, where, Q is the demand, P is the price of the commodity and I is consumer's income.

Find, i) the partial price elasticity of demand,
ii)the partial income elasticity of demand,
iii) evaluate the elasticities at $P=25$ and $I=5000$, also comment on the economic nature of the commodity with respect to consumer's income.

$$Q = 700 - 2p + 0.02I$$

$$E_d = \frac{\partial Q}{\partial P} * \frac{P}{Q} \Rightarrow E_d = -2 * \frac{P}{Q}$$

$$E_I = \frac{\partial Q}{\partial I} * \frac{I}{Q} \Rightarrow E_I = 0.02 * \frac{I}{Q}$$

$$P = 25, I = 5000 \Rightarrow Q = 750$$

$$\Rightarrow E_d = -2 * \frac{25}{750} \Rightarrow E_d = 0.0714 \text{ inelastic}$$

$$\Rightarrow E_I = 0.02 * \frac{5000}{750} \Rightarrow E_d = 0.1333 \text{ normal good}$$

5. Find the optimum level of capital and labour input in order to maximise the following Cobb-Douglas production function, subject to the production constraint, using Lagrange multipliers.

$$Q = L^{0.5} K^{0.3} \quad K + L = 384 \quad \Rightarrow \quad 384 - K - L = 0$$

$$L^* = L^{0.5} K^{0.3} + \lambda(384 - K - L)$$

$$\text{FOC: } \frac{\partial L^*}{\partial L} = \frac{K^{3/10}}{2L^{1/2}} - \lambda = 0$$

$$\frac{\partial L^*}{\partial K} = \frac{3L^{1/2}}{10K^{7/10}} - \lambda = 0 \quad \Rightarrow \quad K = 144, \quad L = 240, \quad \lambda = 0.143$$

$$\frac{\partial L^*}{\partial \lambda} = 384 - K - L = 0$$

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & -\frac{1}{4}K^{0.3}L^{-1.5} & \frac{3}{20}K^{-0.7}L^{-0.5} \\ 1 & \frac{3}{20}K^{-0.7}L^{-0.5} & -\frac{21}{100}L^{0.5}K^{-1.7} \end{vmatrix} \Rightarrow \text{when } K = 144, L = 240 \text{ and } \lambda = 0.143$$

$$\Rightarrow \begin{vmatrix} 0 & 1 & 1 \\ 1 & -0.0003 & 0.0002 \\ 1 & 0.0002 & -0.0007 \end{vmatrix} = 0.0014 > 0 \quad \text{Maximum}$$

6. Given the following total cost function for a firm, find the optimum level of production for the goods, x and z, which minimises the total cost, subject to the capacity restriction, using Lagrange multipliers.

$$TC = 6Q_1^2 + 10Q_2^2 - Q_1Q_2 + 30 \quad Q_1 + Q_2 = 34$$

$$L = 6Q_1^2 + 10Q_2^2 - Q_1Q_2 + 30 + \lambda(34 - Q_1 - Q_2)$$

$$\frac{\partial L}{\partial Q_1} = 12Q_1 - Q_2 - \lambda = 0$$

$$\frac{\partial L}{\partial Q_2} = 20Q_2 - Q_1 - \lambda = 0 \quad \Rightarrow \quad Q_1 = 21, \quad Q_2 = 13, \quad \lambda = 239$$

$$\frac{\partial L}{\partial \lambda} = 34 - Q_1 - Q_2 = 0$$

$$HB = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 12 & -1 \\ 1 & -1 & 20 \end{vmatrix} \Rightarrow |HB_3| = -34 < 0 \quad \therefore \text{minimum is confirmed}$$